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Complex Large-Scale Convection of a Viscous Incompressible Fluid with Heat Exchange According to Newton's Law

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Abstract. The paper considers the construction of analytical solutions to the Oberbeck-Boussinesq system. This system describes layered Bénard-Marangoni convective flows of an incompressible viscous fluid. The third-kind boundary condition, i. e. Newton's heat transfer law, is used on the boundaries of a fluid layer. The obtained solution is analyzed. It is demonstrated that there is a fluid layer thickness with tangential stresses vanishing simultaneously, this being equivalent to the existence of tensile and compressive stresses.

INTRODUCTION

The paper studies the analytical solution describing complex stationary convection in view of the thermocapillary Marangoni effect [1-5], with heat transfer at the boundaries of the flow obeying the Newton-Richman law [6, 7]. The analysis of the solution is made. It is shown that, with certain combinations of the fluid parameters, there appear layers with opposite directions of velocity and layers with zero friction force.

A characteristic feature of the solution is the one-dimensionality of the velocities on coordinates and the three-dimensionality of the temperature and pressure fields. The selected conditions correspond to the theoretical and experimental studies on hydrodynamics [1–13].

For layered flows to be implemented, the condition $V_z = 0$ [1–12] must be satisfied. It is assumed that the buoyant force $\rho_0 g\beta T$ for convection is balanced by the pressure gradient in the vertical direction. The general system of differential equations describing layered stationary convection in an incompressible fluid in the Oberbeck-Boussinesq approximation, with $V_z = 0$, has the form [1–12]

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu\Delta V_x, \quad V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{\partial P}{\partial y} + \nu\Delta V_y,$$

$$\frac{\partial P}{\partial z} = g\beta T, \quad V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} = \chi\Delta T, \quad \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \tag{1}$$

The following standard notation is used in system (1): χ is thermal diffusivity; ν is the coefficient of kinematic viscosity of the fluid; β is the coefficient of fluid volume expansion; g is free fall acceleration; V_x, V_y are fluid velocity vector components.

We define the boundary conditions. Since the fluid is viscous, the adhesion conditions

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$$V_x(x,y,0)=0,\quad V_y(x,y,0)=0 \tag{2}$$

are met at the lower (non-deformable) surface for velocity.

On the free surface defined by the equation $z=h$, the conditions of thermocapillary convection [6,7] are set as

$$\eta\left.\frac{\partial V_x}{\partial z}\right|_{z=h}=-\sigma\left.\frac{\partial T}{\partial y}\right|_{z=h},\ \eta\left.\frac{\partial V_y}{\partial z}\right|_{z=h}=-\sigma\left.\frac{\partial T}{\partial x}\right|_{z=h}, \tag{3}$$

where $\sigma>0$ is the temperature coefficient of surface tension; η is the dynamic viscosity of the fluid [6,7].

At the upper boundary $z=h$ and the lower surface $z=0$, the conditions of heat transfer are specified according to the Newton-Richman law [13] as

$$\lambda\left.\frac{\partial T}{\partial n}\right|_{z=h}=\alpha_1\left(T_{C1}(x,y)-T(x,y,h)\right),\ \lambda\left.\frac{\partial T}{\partial n}\right|_{z=0}=\alpha_2\left(T_{C2}(x,y)-T(x,y,0)\right). \tag{4}$$

$T_{C1}(x,y)=A_1x+B_1y$ is environmental temperature; $T_{C2}(x,y)=A_2x+B_2y$ is the temperature of the lower surface; A_1, B_1, A_2 and B_2 are the temperature gradient components for the free and rigid boundaries, respectively; α_1 and α_2 are the coefficients of heat transfer. The pressure normalized to the density of the fluid on the external free surface has the form $P(x,y,h)=S=s/\rho_0$, where s is atmospheric pressure, ρ_0 is fluid density.

THE ANALYSIS OF THE SOLVABILITY OF THE OBERBECK-BOUSSINESQ SYSTEM FOR LAYERED CONVECTION

The system of equations (1) is overdetermined. Thus, for the description of complex convection, the structure of the solution class must be such that there is one “extra” equation guaranteeing problem solvability. The stationary solution is sought in the form reported in [2–9],

$$V_x=U(z), V_y=V(z),\ T(x,y,z)=T_0(z)+xT_1(z)+yT_2(z),\ P(x,y,z)=P_0(z)+xP_1(z)+yP_2(z). \tag{5}$$

The plane-parallel motion described by the overdetermined Oberbeck-Boussinesq system within the class of solutions (5) was first presented in [2, 3]. Analytical expressions are described there for various types of boundary conditions used in mathematical modeling of advective flows.

Substituting expression (5) for the hydrodynamic fields in equation (1), we obtain the following system of ordinary differential equations to determine the form of the unknown functions (5), which depend on the vertical coordinate z:

$$\begin{gathered}\frac{\mathrm{d}^2T_1}{\mathrm{d}z^2}=0,\quad \frac{\mathrm{d}^2T_2}{\mathrm{d}z^2}=0,\quad \frac{\mathrm{d}P_1}{\mathrm{d}z}=g\beta T_1,\quad \frac{\mathrm{d}P_2}{\mathrm{d}z}=g\beta T_2,\\ \nu\frac{\mathrm{d}^2U}{\mathrm{d}z^2}=P_1,\quad \nu\frac{\mathrm{d}^2V}{\mathrm{d}z^2}=P_2,\quad \frac{\mathrm{d}^2T_0}{\mathrm{d}z^2}=\frac{1}{\chi}\left(UT_1+VT_2\right),\quad \frac{\mathrm{d}P_0}{\mathrm{d}z}=g\beta T_0.\end{gathered} \tag{6}$$

The equations in system (6) are written in the order in which integration will be performed. Due to the structure of expressions (5), the boundary conditions (2)-(4) are written as follows:

$$
\begin{aligned}
&\left.\frac{\mathrm{d}T_1}{\mathrm{d}z}\right|_{z=0} = -\vartheta_1\left(A_1 - T_1(0)\right), \quad \left.\frac{\mathrm{d}T_2}{\mathrm{d}z}\right|_{z=0} = \vartheta_2\left(B_1 - T_2(0)\right),\\
&\left.\frac{\mathrm{d}T_1}{\mathrm{d}z}\right|_{z=h} = \vartheta_2\left(A_2 - T_1(h)\right), \quad \left.\frac{\mathrm{d}T_2}{\mathrm{d}z}\right|_{z=h} = \vartheta_2\left(B_2 - T_2(h)\right),\\
&T_0(x,y,h)=0, \quad T_0(x,y,0)=0, \quad U(0)=0, \quad V(0)=0.
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\eta\left.\frac{\mathrm{d}U}{\mathrm{d}z}\right|_{z=h} = -\sigma T_1(h), \quad \eta\left.\frac{\mathrm{d}V}{\mathrm{d}z}\right|_{z=h} = -\sigma T_2(h),\\
&P_1(h)=0, \quad P_2(h)=0, \quad P_0(h)=S.
\end{aligned}
\tag{8}
$$

where $\vartheta_1 = \alpha_1 h/\lambda$, $\vartheta_2 = \alpha_2 h/\lambda$ is the Nusselt number for the upper and lower boundaries, respectively.

It is obvious that the continuity equation involved in system (1) is fulfilled identically for this choice of solution representation.

EXACT SOLUTION OF THE BOUNDARY VALUE PROBLEM

The solution for the first two equations (6) defining the functions T_1 and T_2 is sought in the form of linear functions as

$$
T_1 = a_1\frac{z}{h} + b_1, \quad T_2 = a_2\frac{z}{h} + b_2.
$$

The coefficients of the solution satisfying the boundary conditions (7) are found from calculations as follows:

$$
a_1 = \frac{\left(A_2 - A_1\right)\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}, \quad b_1 = \frac{A_1\vartheta_1 + A_2\vartheta_2 + A_1\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}, \quad a_2 = \frac{\left(B_2 - B_1\right)\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}, \quad b_2 = \frac{B_1\vartheta_1 + B_2\vartheta_2 + B_1\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}.
$$

The components of the gradient of reduced pressure have a parabolic profile. The velocity of convective fluid motion is determined by polynomials, quartic in the general case, as

$$
\begin{aligned}
U &= -\frac{\sigma h}{\eta}\left(a_1+b_1\right)\frac{z}{h} + \frac{g\beta}{\nu}h^3\left(\frac{a_1}{24}\frac{z^4}{h^4} + \frac{b_1}{6}\frac{z^3}{h^3} - \frac{a_1+2b_1}{4}\frac{z^2}{h^2} + \frac{2a_1+3b_1}{6}\frac{z}{h}\right),\\
V &= -\frac{\sigma h}{\eta}\left(a_2+b_2\right)\frac{z}{h} + \frac{g\beta}{\nu}h^3\left(\frac{a_2}{24}\frac{z^4}{h^4} + \frac{b_2}{6}\frac{z^3}{h^3} - \frac{a_2+2b_2}{4}\frac{z^2}{h^2} + \frac{2a_2+3b_2}{6}\frac{z}{h}\right).
\end{aligned}
$$

DETERMINATION OF ZERO-FRICTION LAYER THICKNESS

Let the components of the gradient vectors be related as

$$
A_2 = \gamma_x A_1 \text{ and } B_2 = \gamma_y B_1, \tag{9}
$$

where γ_x and γ_y are constants. We now calculate the tangential stresses τ_{zx}, τ_{zy} in the fluid layer and consider the equations

$$
\tau_{zx} = 0, \quad \tau_{zy} = 0. \tag{10}
$$

After the substitution of the gradient relationships (9) and some transformations, equations (10) become

$$\frac{g\beta\rho}{6}h^2\left\{\frac{(\gamma_x-1)\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\left(\frac{z}{h}\right)^3+3\frac{\vartheta_1+\gamma_x\vartheta_2+\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\left(\frac{z}{h}\right)^2-3\frac{(\gamma_x+1)\vartheta_1\vartheta_2+2(\vartheta_1+\gamma_x\vartheta_2)}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\frac{z}{h}\right.$$
$$\left.+\frac{(2\gamma_x+1)\vartheta_1\vartheta_2+3(\vartheta_1+\gamma_x\vartheta_2)}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\right\}-\sigma\frac{\vartheta_1+\gamma_x\vartheta_2+\gamma_x\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}=0, \tag{11}$$

$$\frac{g\beta\rho}{6}h^2\left\{\frac{(\gamma_y-1)\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\left(\frac{z}{h}\right)^3+3\frac{\vartheta_1+\gamma_y\vartheta_2+\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\left(\frac{z}{h}\right)^2-3\frac{(\gamma_y+1)\vartheta_1\vartheta_2+2(\vartheta_1+\gamma_y\vartheta_2)}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\frac{z}{h}+\right.$$
$$\left.+\frac{(2\gamma_y+1)\vartheta_1\vartheta_2+3(\vartheta_1+\gamma_y\vartheta_2)}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\right\}-\sigma\frac{\vartheta_1+\gamma_y\vartheta_2+\gamma_y\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}=0. \tag{12}$$

Let us now find the common roots of equations (11) and (12) with respect to z/h. The left-hand side of equation (12) is subtracted from the left-hand side of equation (11). The result is

$$(\gamma_x-\gamma_y)\left\{\frac{g\beta\rho}{6}h^2\left[\frac{\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\left(\frac{z}{h}\right)^3+3\frac{\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\left(\frac{z}{h}\right)^2-\right.\right.$$
$$\left.\left.-3\frac{(2\vartheta_2+\vartheta_1\vartheta_2)}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\frac{z}{h}+\frac{3\vartheta_2+2\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\right]-\sigma\frac{\vartheta_2+\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2+\vartheta_1\vartheta_2}\right\}=0.$$

If $\gamma_x=\gamma_y=\gamma,$ the right-hand sides of equations (11) and (12) coincide and the equations have identical roots. The relative depth $\delta=z/h$ is introduced. The parameter δ can assume values in the range $0<\delta<1,$ where the value 0 corresponds to the solid surface and 1 corresponds to the free surface. The equation for layer thickness takes the form

$$h^2=\frac{6\sigma}{g\beta\rho}(\vartheta_1+\gamma\vartheta_2+\gamma\vartheta_1\vartheta_2)\left\{(\gamma-1)\vartheta_1\vartheta_2\delta^3+3(\vartheta_1+\gamma\vartheta_2+\gamma\vartheta_1\vartheta_2)\delta^2-\right.$$
$$\left.-3\left[(\gamma+1)\vartheta_1\vartheta_2+2(\vartheta_1+\gamma\vartheta_2)\right]\delta+\left[(2\gamma+1)\vartheta_1\vartheta_2+3(\vartheta_1+\gamma\vartheta_2)\right]\right\}^{-1}. \tag{13}$$

The thickness h can then be represented in the form

$$h=\frac{1}{1-\delta}\sqrt{\frac{6\sigma(\vartheta_1+\gamma\vartheta_2+\gamma\vartheta_1\vartheta_2)}{g\beta\rho\left[3(\vartheta_1+\gamma\vartheta_2)+\vartheta_1\vartheta_2(1+2\gamma)+\delta(\gamma-1)\vartheta_1\vartheta_2\right]}}. \tag{14}$$

The parameter σ is positive for most fluids. The following variants of the parameter values are possible:

1. when $\gamma\geq 1,$ the denominator of the radicand in (14) and the parenthesized expression in the numerator are positive for all the admissible values of the component parameters; therefore, the layer thickness at which there is a zero-friction layer exists for all admissible parameter values;
2. when $\gamma<1$, the following two cases are possible:

a) the inequalities

$$\vartheta_1+\gamma\vartheta_2+\gamma\vartheta_1\vartheta_2\geq 0 \tag{15}$$

and

$$3(\vartheta_1+\gamma\vartheta_2)+\vartheta_1\vartheta_2(1+2\gamma)+\delta(\gamma-1)\vartheta_1\vartheta_2>0 \tag{16}$$

are fulfilled; it follows from inequality (15) that $-\frac{\vartheta_1}{\vartheta_2(1+\vartheta_1)} \le \gamma < 1$ and from inequality (16) that $\delta < \frac{3(\vartheta_1+\gamma\vartheta_2)+\vartheta_1\vartheta_2(1+2\gamma)}{(1-\gamma)\vartheta_1\vartheta_2}$, and, in view of the restrictions on the parameter δ,

$$0 \le \delta < \min\left\{\frac{3(\vartheta_1+\gamma\vartheta_2)+\vartheta_1\vartheta_2(1+2\gamma)}{(1-\gamma)\vartheta_1\vartheta_2}, 1\right\};$$

however, the fraction inside the braces exceeds 1, and the inequality $0 \le \delta < 1$ is obtained for δ, consequently, for any admissible value of δ there is such layer thickness h that the friction forces at the depth δh will be zero;

b) the parameters ϑ_1, ϑ_2, δ satisfy the inequalities

$$\vartheta_1 + \gamma\vartheta_2 + \gamma\vartheta_1\vartheta_2 < 0 \tag{17}$$

and

$$3(\vartheta_1+\gamma\vartheta_2)+\vartheta_1\vartheta_2(1+2\gamma)+\delta(\gamma-1)\vartheta_1\vartheta_2 < 0. \tag{18}$$

It follows from (17) that the parameter γ must satisfy the inequality $\gamma < -\frac{\vartheta_1}{\vartheta_2(1+\vartheta_1)}$, and, from (18), it can be written for δ that $\max\left\{0, \frac{3(\vartheta_1+\gamma\vartheta_2)+\vartheta_1\vartheta_2(1+2\gamma)}{(1-\gamma)\vartheta_1\vartheta_2}\right\} \le \delta < 1$. The inequality $\frac{3(\vartheta_1+\gamma\vartheta_2)+\vartheta_1\vartheta_2(1+2\gamma)}{(1-\gamma)\vartheta_1\vartheta_2} < 1$ must be fulfilled for the existence of the nonempty range of the δ values. It is easy to verify that this inequality is equivalent to inequality (18), and that it is fulfilled in the parameter value range defined by inequalities (17) and (18). Consequently, there is a nonempty range of the values of δ, and, at any value of δ from this range, there is such layer thickness h that the friction forces at the depth δh will be zero.

It should be noted that, under conditions of a perfect thermal contact on the both fluid layer boundaries, equations (10) turn into the quadratic equations with equal roots $\pm\sqrt{\frac{6\gamma\sigma}{g\beta\rho(1+2\gamma)}}$. The root with the + sign takes actual values when $\gamma < -1/2$ or $\gamma > 0$.

CONCLUSION

The paper has discussed stationary layered convective flows of a viscous incompressible fluid, which are induced by a temperature gradient. Solutions have been obtained for the third-kind boundary conditions on the fluid flow boundaries, i. e. here we deal with heat transfer by Newton's law. It has been demonstrated that, under certain conditions, there is a fluid layer thickness at which the tangential stresses on the solid surface vanish simultaneously.

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